Cours du 12 novembre 2026

Ex 7 p 29

1) $\forall x \in \mathbb{R}$ $\int_0^x t e^t \, dt$

$\forall t \in [0,x]$ $u(t) = t$ $\quad u'(t) = 1$ $\quad v(t) = e^t$ $\quad v'(t) = e^t$

$u, v \in \mathcal{C}^1([0,x])$

par théorème d'intégration par parties,

$$\int_0^x t e^t \, dt = \left[t e^t \right]_0^x - \int_0^x e^t \, dt = x e^x - \left[e^t \right]_0^x$$

$$= x e^x - e^x + 1$$

2) Déjà vu

3) $\int_0^1 1 \times \operatorname{Arctan}(t) \, dt$

$\forall x \in [0,1]$ on pose

$u(t) = \operatorname{Arctan}(t)$ $\quad u'(t) = \frac{1}{1+t^2}$ $\quad v(t) = t$ $\quad v'(t) = 1$

$(u, v) \in \mathcal{C}^1([0,1])$

par intégration par parties,

$$\int_0^1 \operatorname{Arctan}(t) \, dt = \left[t \operatorname{Arctan}(t) \right]_0^1 - \int_0^1 \frac{t}{1+t^2} \, dt$$

$$= \operatorname{Arctan}(1) - \frac{1}{2} \left[\ln(1+t^2) \right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

4) $\int_0^\pi t \cos(t) \, dt$ $\longrightarrow$ polynôme - exp. $\quad \operatorname{Re}\left(\int_0^\pi t e^{it} \, dt \right)$

sous la forme $f(t) e^{it}$ avec $f(t) = at + b$

$f'(t) = (ait + bi + a) e^{it}$

on prend $a = \frac{1}{i} = -i$

et $bi = -a = i$

donc $b = 1$

donc $f : t \mapsto (-it+1)e^{it}$

$f \in \mathcal{D}'(\mathbb{R})$, $\forall t \in \mathbb{R}$, $f'(t) = te^{it}$

donc $\int_0^\pi t\cos(t)\,dt = \mathrm{Re}\left(\left[(-it+1)e^{it}\right]_0^\pi \right)$

$= \mathrm{Re}(-(i\pi+1) - 1)$

$= -2$

intégration par partie

$\int_0^\pi t\cos(t)\,dt$ $\quad \forall t \in [0,\pi]$ on pose $u(t) = t$, $v(t) = \sin(t)$

$u'(t) = 1$, $v'(t) = \cos(t)$

$u, v \in \mathcal{C}^1([0,\pi])$

par intégration par parties.

$\int_0^\pi t\cos(t)\,dt = \left[t\sin(t)\right]_0^\pi - \int_0^\pi \sin(t)\,dt$

$= \pi\sin\pi - 0 + \left[\cos(t)\right]_0^\pi$

$= \cos\pi - \cos(0) = -1 - 1 = -2$

ex II p.30

$\int^x \frac{dt}{2t^2+5t+2}$

$\Delta = 25 - 16$

$= 9$

$= \int^x \frac{1\,dt}{2(t+2)(t+\frac{1}{2})} = \frac{1}{2}\cdot\frac{2}{3}\int^x \frac{t+2-(t+\frac{1}{2})}{(t+2)(t+\frac{1}{2})}\,dt$

$= \frac{1}{3}\int^x \frac{1}{t+\frac{1}{2}} - \frac{1}{t+2}\,dt$

$= \frac{1}{3}\left(\ln\left|x+\frac{1}{2}\right| - \ln|x+2|\right)$

$= \frac{1}{3}\ln\left|\frac{x+\frac{1}{2}}{x+2}\right| = \frac{1}{3}\ln\left|\frac{2x+1}{2x+4}\right|$

4) $\int^x \frac{t}{t^4+2t^2+3}\,dt$ on pose $u = t^2$ $du = 2t\,dt$ $u \in \mathcal{C}^1(\mathbb{R})$

$$\frac{1}{2}\int^{x} \frac{2t\,dt}{t^4+2t^2+3} = \frac{1}{2}\int^{x^2} \frac{dv}{v^2+2v+3} = \frac{1}{2}\int^{x^2} \frac{dv}{(v+1)^2+2}$$

$$= \frac{1}{4}\int^{x^2} \frac{dv}{\left(\frac{v+1}{\sqrt{2}}\right)^2+1} \qquad u = \frac{v+1}{\sqrt{2}} \qquad du = \frac{dv}{\sqrt{2}}$$

$$v \in \mathcal{C}^1(\mathbb{R})$$

$$= \frac{1}{4}\int^{\frac{x^2+1}{\sqrt{2}}} \frac{\sqrt{2}\,du}{u^2+1}$$

$$= \frac{\sqrt{2}}{4}\arctan\left(\frac{x^2+1}{\sqrt{2}}\right)$$

Ex 12

$$\int^{x} \frac{dt}{1+\cos(t)}$$

$u = \cos(t)$ $\quad du = -\sin(t)\,dt$

$u = \sin(t)$ $\quad du = \cos(t)\,dt$

$$= \int^{x} \frac{dt}{1+\frac{1-\tan^2(t/2)}{1+\tan^2(t/2)}}$$

$$\cos(t) = \cos^2(t/2) - \sin^2(t/2) = \cos^2(t/2)\left(1-\tan^2(t/2)\right)$$
$$= \frac{1-\tan^2(t/2)}{1+\tan^2(t/2)}$$

$$= \int^{x} \frac{\left(1+\tan^2(t/2)\right)dt}{1+\tan^2(t/2)+1-\tan^2(t/2)}$$

$$= \int^{x} \frac{1+\tan^2(t/2)}{2}\,dt = \left[\tan(t/2)\right]^{x} = \tan(x/2)$$

2) $$\int^{x} \frac{dt}{2+\cos(t)} = \int^{x} \frac{\left(1+\tan^2(t/2)\right)}{2+2\tan^2(t/2)+1-\tan^2(t/2)}\,dt$$

$$= \int^{x} \frac{1+\tan^2(t/2)}{3+\tan^2(t/2)}\,dt$$

$u = \tan(t/2)$ $\quad du = \frac{1}{2}\left(1+\tan^2(t/2)\right)dt$

$$= 2\int^{\tan x/2} \frac{du}{3+u^2}$$

$$= \frac{2}{3}\int^{\tan x/2} \frac{du}{1+\left(u/\sqrt{3}\right)^2}$$

on pose $v = u/\sqrt{3}$

$dv = \frac{du}{\sqrt{3}}$

donc $du = \sqrt{3}\,dv$

$$= \frac{2}{3}\int^{\frac{\tan(x/2)}{\sqrt{3}}} \frac{\sqrt{3}\,dv}{1+v^2}$$

$$= \frac{2\sqrt{3}}{3}\,\mathrm{Arctan}\left(\frac{\tan(x/2)}{\sqrt{3}}\right)$$

ex 13 1) $\int^x \frac{th(t)}{1+th(t)}\,dt = \int^x \frac{1+th(t)-1}{1+th(t)}$?

$$1 - \frac{1}{1+th\,t}$$

$$\int^x \frac{\frac{e^t-e^{-t}}{e^t+e^{-t}}}{1+\frac{e^t-e^{-t}}{e^t+e^{-t}}}\,dt = \int^x \frac{e^t-e^{-t}}{2e^t}\,dt$$

$$= \int^x \frac{1}{2} - \frac{e^{-2t}}{2}\,dt$$

$$= \frac{x}{2} + \frac{1}{4}\left[e^{-2t}\right]^x$$

$$= \frac{x}{2} + \frac{e^{-2x}}{4}$$

Ex 14

1) $f : x \mapsto \int_{-x}^{x} \frac{dt}{1+t^4}$ $\qquad t \mapsto \frac{1}{1+t^4} \in \mathcal{C}^0(\mathbb{R})$

on pose F une primitive de $t \mapsto \frac{1}{1+t^4}$ sur $\mathbb{R}$.

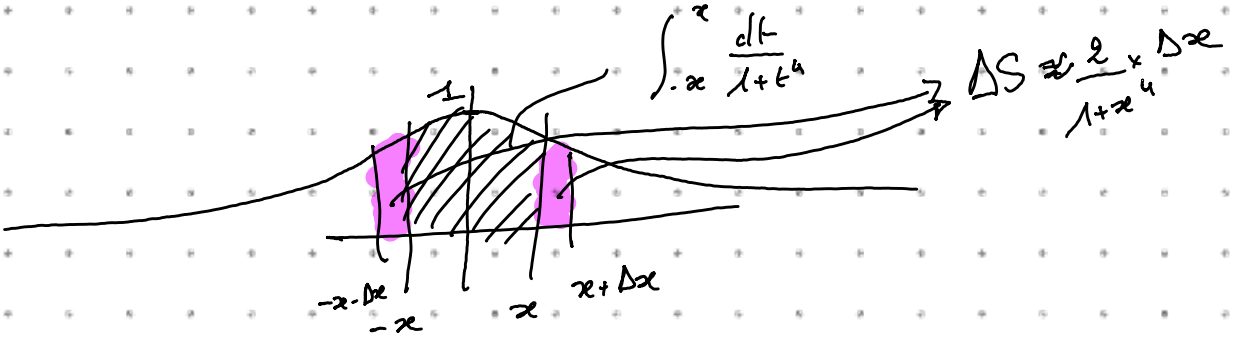
$\forall x \in \mathbb{R}, \quad f(x) = \left[F(t)\right]_{-x}^{x} = F(x) - F(-x)$

$F \in \mathcal{D}^1(\mathbb{R})$

donc par composition et somme $f \in \mathcal{D}^1(\mathbb{R})$

$\forall x \in \mathbb{R} \quad f'(x) = F'(x) + F'(-x)$

$= \frac{1}{1+x^4} + \frac{1}{1+(-x)^4} = \frac{2}{1+x^4}$



2) $f(x) = \int_0^{2\pi} x \cos^5(tx)\, dt = \int_0^{2\pi} \cos^5(tx) \cdot x\, dt$ $\quad u = tx$ $\quad$ à x fixé

$du = x\, dt$

$u \in \mathcal{C}^1([0, 2\pi x])$

$= \int_0^{2\pi x} \cos^5(u)\, du$

$= F(2\pi x)$ où $F : x \mapsto \int_0^x \cos^5(u)\, du$

$F \in \mathcal{D}^1(\mathbb{R})$

donc par composition $f \in \mathcal{D}^1(\mathbb{R})$

$\forall x \in \mathbb{R}, \quad f'(x) = 2\pi F'(2\pi x) = 2\pi \cos^5(2\pi x)$

Ex 15

$\int_0^x e^t \cos(t)\, dt$ ⟶ polynome-exponentielle

$\mathrm{Re}\left(\int_0^x e^t e^{it}\, dt\right)$

$= \mathrm{Re}\left(\int_0^x e^{(1+i)t}\, dt\right)$

$= \mathrm{Re}\left(\left[\frac{1}{1+i} e^{(1+i)t}\right]_0^x\right)$

$= \mathrm{Re}\left(\left[\frac{1-i}{2} e^t \times e^{it}\right]_0^x\right)$

$$= \left[\frac{1}{2} e^t \operatorname{Re}\left((1-i)e^{it}\right) \right]_0^x$$

$$= \left[\frac{1}{2} e^t \left(\cos(t) + \sin t \right) \right]_0^x$$

$$= \frac{1}{2} e^x (\cos x + \sin x) - \frac{1}{2} \quad \leftarrow$$

Double intégration par partie.

$$\int_0^x e^t \cos(t)\, dt \qquad \forall t \in [0,x], \quad u(t) = \cos(t) \quad — \quad v(t) = e^t$$
$$u'(t) = -\sin(t) \quad \backslash \quad v'(t) = e^t$$
$$(u,v) \in \mathcal{C}^1([0,x])$$

$$\int_0^x e^t \cos(t)\, dt = \left[\cos(t) e^t \right]_0^x + \int_0^x \sin(t) e^t\, dt$$

$$= (\cos x) e^x - 1 + \int_0^x \sin(t) e^t\, dt$$

$\forall t \in [0,x]$ on pose

$$u(t) = \sin(t) \quad \backslash \quad v(t) = e^t$$
$$u'(t) = \cos(t) \quad v'(t) = e^t$$
$$(u,v) \in \mathcal{C}^1([0,x])$$

$$\int_0^x e^t \cos(t)\, dt = (\cos x) e^x - 1 + \left[\sin(t) e^t \right]_0^x - \int_0^x \cos(t) e^t\, dt$$

$$I = (\cos x) e^x - 1 + (\sin x) e^x - \int_0^x \cos(t) e^t\, dt$$

I $\qquad$ I

$$2I = (\cos x + \sin x) e^x - 1$$

donc $$\int_0^x e^t \cos(t)\, dt = I = \frac{1}{2}\left((\cos x + \sin x) e^x - 1 \right)$$

6) $\forall x > -1$ $\qquad \int_0^x e^{\sqrt{1+t}}\, dt \qquad u = \sqrt{1+t} \qquad du = \frac{1}{2\sqrt{1+t}}\, dt$

$u \in \mathcal{C}^1([0,x])$

$$dt = 2\sqrt{1+t}\,du$$
$$= 2u\,du$$

$$\int_0^x e^{\sqrt{1+t}}\,dt = \int_1^{\sqrt{1+x}} e^u \, 2u\,du$$

$$= 2\int_1^{\sqrt{1+x}} u e^u\,du$$

$$= 2\Big[t e^t - e^t\Big]_1^{\sqrt{1+x}}$$

$$= 2\left(\sqrt{1+x}-1\right)e^{\sqrt{1+x}}$$

Ex 21 p.30 $\quad \forall n \in \mathbb{N} \quad I_n = \int_0^{\pi/2} \sin^n(t)\,dt$

$$I_{n+2} = \int_0^{\pi/2} \sin^{n+2}(t)\,dt$$

$$= \int_0^{\pi/2} \underbrace{\sin(t)}_{\uparrow} \times \underbrace{\sin^{n+1}(t)}_{\downarrow}\,dt$$

$\forall t \in [0, \pi/2]$, on pose

$u(t) = \sin^{n+1}(t)$ $\qquad$ $v(t) = -\cos(t)$

$u'(t) = (n+1)\cos(t)\sin^n(t)$ $\qquad$ $v'(t) = \sin(t)$

$u, v \in \mathcal{C}^1([0, \pi/2])$

par intégration par parties :

$$I_{n+2} = \underbrace{\Big[-\cos(t)\sin^{n+1}(t)\Big]_0^{\pi/2}}_{=0} + (n+1)\int_0^{\pi/2} \cos^2(t)\sin^n(t)\,dt$$

$$= (n+1)\int_0^{\pi/2}\left(1-\sin^2(t)\right)\sin^n(t)\,dt$$

$$= (n+1)\left[\int_0^{\pi/2}\sin^n(t)\,dt - \int_0^{\pi/2}\sin^{n+2}(t)\,dt\right]$$

donc $\quad I_{n+2} = (n+1)\,I_n - (n+1)\,I_{n+2}$

donc $\quad (n+2)\,I_{n+2} = (n+1)\,I_n$

$$\forall n \in \mathbb{N}, \quad I_{n+2} = \frac{n+1}{n+2} I_n .$$

Réf $I_0 = \int_0^{\pi/2} \sin^0(t)\,dt = \pi/2$ $\qquad I_1 = \int_0^{\pi/2} \sin(t)\,dt = [-\cos t]_0^{\pi/2} = 1$

pour $n = 2p$, $p \in \mathbb{N}$

$$I_{2p} = \frac{2p-1}{2p} I_{2(p-1)} = \frac{2p-1}{2p} \times \frac{2p-3}{2(p-1)} \times I_{2(p-2)}$$

$$\vdots$$

$$= \frac{2p-1}{2p} \times \frac{2p-3}{2(p-1)} \times \dots \times \frac{3}{4} \times \frac{1}{2} I_0$$

$I_{n+2} = \frac{n+1}{n+2} I_n$

$$I_2 = \frac{1}{2} I_0$$

$$I_4 = \frac{3}{4} I_2$$

$$= \frac{3}{4} \times \frac{1}{2} I_0$$

$$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} I_0$$

$$I_8 = \frac{7}{8} I_6 = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} I_0$$

$$I_{2p} = \frac{(2p-1) \times \dots \times 5 \times 3 \times 1}{(2p \times \dots \times 6 \times 4 \times 2)^2} I_0$$

$$= \frac{(2p)!}{(2p \times 2(p-1) \times 2(p-2) \cdots \times 2 \times 3 \times 2 \times 2 \times 2 \times 1)^2} \times \pi/2$$

$$= \frac{(2p)!}{(2^p \cdot p!)^2} \pi/2$$

$$I_3 = \frac{2}{3} \times I_1$$

$$I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} I_1$$

$$I_7 = \frac{6}{7} I_5 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \underbrace{I_1}_{=1}$$

$$I_{2p+1} = \frac{\left[(2p) \times \cdots \cdots 6 \times 4 \times 2\right]^2}{(2p+1) \cdots \cdots \times 5 \times 3 \times 1} \times 1$$

($\times 2p$; $\times 6$; $\times 4$; $\times 2$)

$$= \frac{\left(2^p \times p!\right)^2}{(2p+1)!}$$

$$I_n = \int_0^{\pi/2} \cos^n(t)\, dt \qquad u = \pi/2 - t, \qquad du = -dt$$

$$u \in \mathcal{C}^1([0, \pi/2])$$

$$t = \pi/2 - u$$

$$= \int_{\pi/2}^{0} \cos^n\left(\frac{\pi}{2} - u\right)(-du)$$

$$= -\int_0^{\pi/2} \sin^n(u) \times (-du) = \int_0^{\pi/2} \sin^n(u)\, du = I_n .$$