TD du 10/11/2026

Ex 1 p. 29

$\int_0^x te^t\,dt =$

$\forall t \in [0,x]$, $u(t) = t$ ——— $v(t) = e^t$

$u'(t) = 1$ ——— $v'(t) = e^t$

$u, v \in \mathcal{C}^1([0,x])$

par intégration par parties :

$$\int_0^x te^t\,dt = \left[te^t\right]_0^x - \int_0^x e^t\,dt = xe^x - \left[e^t\right]_0^x = xe^x - e^x + 1$$
$$= (x-1)e^x + 1$$

Autre approche : polynôme × exponentielle.

chercher une primitive sous la forme $f(t) = (at+b)e^t$

$f \in \mathcal{C}^1([0,x])$, $\forall t \in [0,x]$, $f'(t) = (at + a + b)e^t$

$$\begin{cases} a = 1 \\ a + b = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = -1 \end{cases}$$

si on pose $f : t \mapsto (t-1)e^t$ c'est une primitive de $t \mapsto te^t$

$$\int_0^x te^t\,dt = \left[(t-1)e^t\right]_0^x = (x-1)e^x + 1$$

2) $\int_1^e t^2 \ln(t)\,dt$

$\forall t \in [1,e]$, $u(t) = \ln(t)$ ——— $v(t) = t^3/3$

$u'(t) = 1/t$ ——— $v'(t) = t^2$

$(u, v) \in \mathcal{C}^1([1,e])$

par intégration par parties

$$\int_1^e t^2 \ln(t)\,dt = \left[\frac{t^3}{3}\ln(t)\right]_1^e - \int_1^e \frac{t^2}{3}\,dt = \frac{e^3}{3} - \left[\frac{t^3}{9}\right]_1^e$$
$$= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2e^3 + 1}{9}$$

Ex 9 p. 29

$$\int_0^{\pi/4} \tan^4(t)\,dt = \int_0^{\pi/4} \left(1 + \tan^2(t)\right)^2 - 2\tan^2(t) - 1\,dt$$
$$= \int_0^{\pi/4} \left(1 + \tan^2(t)\right)^2 - 2\left(\tan^2(t) + 1\right) + 1\,dt$$
$$= \int_0^{\pi/4} \left(1 + \tan^2(t)\right)\left(1 + \tan^2(t) - 2\right) + 1\,dt$$
$$= \int_0^{\pi/4} \left(1 + \tan^2(t)\right)\left(\tan^2(t) - 1\right)dt + \int_0^{\pi/4} 1\,dt$$

$a^2 - b^2$

$\tan^4(t) - 1 + 1$

$= \left(\tan^2(t) - 1\right)\left(\tan^2(t) + 1\right) + 1$

$$= \int_0^{\pi/4} \left(1+\tan^2(t)\right)\left(\tan^2(t)-1\right)dt + \pi/4$$

$u = \tan(t)$

$du = \left(1+\tan^2(t)\right)dt$

$u \in \mathcal{C}^1([0,\pi/4])$

$$= \int_0^1 (u^2-1)\,du + \pi/4$$

$$= \left[\frac{u^3}{3} - u\right]_0^1 + \frac{\pi}{4}$$

$$= 1/3 - 1 + \pi/4 = \pi/4 - 2/3$$

4)

$$\int_0^{\pi/4} \sqrt{1+\sin(t)}\,dt$$

~~$u = \cos(t)$~~

$u = \sin(t)$, $du = \cos(t)\,dt$

$u \in \mathcal{C}^1([0,\pi/4])$

$\sin([0,\pi/4])$ $\cos(t) > 0$

donc $\cos(t) = \sqrt{1-\sin^2(t)}$

$= \sqrt{1-u^2}$

$$dt = \frac{du}{\cos(t)\neq 0} = \frac{du}{\sqrt{1-u^2}}$$

$$\int_0^{\pi/4} \sqrt{1+\sin(t)}\,dt = \int_0^{\sqrt{2}/2} \sqrt{1+u}\,\frac{du}{\sqrt{1-u^2}}$$

$$= \int_0^{\sqrt{2}/2} \sqrt{\frac{1+u}{1-u^2}}\,du$$

$$= \int_0^{\sqrt{2}/2} \sqrt{\frac{1+u}{(1+u)(1-u)}}\,du = \int_0^{\sqrt{2}/2} \frac{du}{\sqrt{1-u}}$$

$w = 1-u$

$$= -2\int_0^{\sqrt{2}/2} \frac{-du}{2\sqrt{1-u}} = -2\left[\sqrt{1-u}\right]_0^{\sqrt{2}/2}$$

$$= -2\sqrt{1-\sqrt{2}/2} + 2$$

$$\sqrt{1+\sin(t)} = \sqrt{1+\frac{2\tan(t/2)}{1+\tan^2(t/2)}}$$

$$= \sqrt{\frac{1+\tan^2(t/2)+2\tan(t/2)}{1+\tan^2(t/2)}} = \sqrt{\frac{\left(1+\tan(t/2)\right)^2}{1+\tan^2(t/2)}}$$

$$= \frac{1+\tan(t/2)}{\sqrt{1+\tan^2(t/2)}}$$

$1+\tan(t/2) > 0$

$u = \tan(t/2)$ $\quad du = \frac{1}{2}(1+\tan^2(t/2))\,dt$

$= \frac{1}{2}(1+u^2)\,dt$

$u \in \mathcal{C}^1([0, \pi/4])$

$$\int_0^{\pi/4} \sqrt{1+\sin(t)}\,dt = \int_0^{\pi/4} \frac{1+\tan(t/2)}{\sqrt{1+\tan^2(t/2)}}\,dt$$

$$= \int_0^{\tan(\pi/8)} \frac{1+u}{\sqrt{1+u^2}} \frac{2}{1+u^2}\,du = 2\int_0^{\tan(\pi/8)} \frac{1+u}{(1+u^2)^{3/2}}\,du$$

$\frac{1}{(1+u^2)^{3/2}}$ Difficile

$\frac{u}{(1+u^2)^{3/2}}$ OK $v = 1+u^2$

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ou $\sin(t) = 2\cos(t/2)\sin(t/2)$

donc $1+\sin(t) = \sin^2(t/2) + \cos^2(t/2) + 2\sin(t/2)\cos(t/2)$

$= \underbrace{(\sin(t/2)+\cos(t/2))^2}_{\geq 0}$

$$\int_0^{\pi/4} \sqrt{1+\sin(t)}\,dt = \int_0^{\pi/4} \sin(t/2)+\cos(t/2)\,dt = \left[-2\cos(t/2)+2\sin(t/2)\right]_0^{\pi/4}$$

$$= -2\cos(\pi/8) + 2(\sin(\pi/8)) + 2$$

$x = \pi/8$ $\quad \cos(2x) = 2\cos^2(x) - 1$

donc pour $x = \pi/8$ $\quad 2\cos^2(\pi/8) - 1 = \cos(\pi/4) = \sqrt{2}/2$

donc $2\cos^2(\pi/8) = \frac{\sqrt{2}}{2} + 1$

donc $\underbrace{\cos(\pi/8)}_{\geq 0} = \sqrt{\frac{1}{2}\left(\sqrt{2}/2 + 1\right)}$

idem pour $\sin(\pi/8)$ $\quad = \sqrt{\frac{\sqrt{2}}{4} + \frac{1}{2}}$

méthode pour calculer des cos, sin sur des "fractions de valeurs remarquables"

cf ex 3 p 11
ex 23 p 20

Ex 10 p. 29

1) $\int_0^{\pi} \cos^3(t)\sin^2(t)\,dt = \int_0^{\pi} \cos(t) \times \cos^2(t)\sin^2(t)\,dt$

$= \int_0^{\pi} \cos(t) \times \left(1-\sin^2(t)\right)\sin^2(t)\,dt$

on pose $u = \sin(t)$

$du = \cos(t)\,dt$:

$u \in \mathcal{C}^1([0,\pi])$

t	0	π
$u = \sin(t)$	$\sin 0 = 0$	$\sin \pi$

$= \int_0^{\sin\pi} (1-u^2)\,u^2\,du$

$= \int_0^{\sin\pi} u^2 - u^4\,du = \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^{\sin\pi}$

$= \frac{\sin^3(\pi)}{3} - \frac{\sin^5(\pi)}{5}$

2) $\int_0^{\pi} \cos^3(t)\sin(t)\,dt = \left[-\frac{\cos^4(t)}{4}\right]_0^{\pi} = \frac{1}{4} - \frac{\cos^4(\pi)}{4}$

$u = \cos(t)$

3) $\sin^4(2t) = \left[2\sin(t)\cos(t)\right]^4$ etc ...

→ Travailler

- fin ex p. 29
- ex 11
- ex 12
- ex 13
- ex 14
- 21

Exercices de cours EDL

$y' - 3y = e^{2t}$

① Equation homogène $y' = 3y$

$\mathcal{S} = \{ t \mapsto \lambda e^{3t},\ \lambda \in \mathbb{R} \}$

② Solution particulière

on ne rédige pas

équation homogène à coefficients constants et second membre

en polynôme x exp.

on cherche une solution $f : t \longmapsto ae^{2t} \in \mathcal{D}'(\mathbb{R})$

$\forall t \in \mathbb{R} \quad f'(t) = 2ae^{2t}$

$e^{2t} = \underbrace{f'(t) - 3f(t)}_{e^{2t} = y' - 3y} = 2ae^{2t} - 3ae^{2t} = -ae^{2t}$

on prend $a = -1$

$t \longmapsto -e^{2t}$ solution particulière

③ Solutions $\mathcal{S} = \{ t \longmapsto \lambda e^{3t} - e^{2t}, \lambda \in \mathbb{R} \}$

Exemple $y' - 3y = xe^{2x}$

① équation homogène $S_0 = \{ x \longmapsto \lambda e^{3x}, \lambda \in \mathbb{R} \}$

② équation particulière pour $(a,b) \in \mathbb{R}^2$, on pose $f : x \longmapsto (ax+b)e^{2x}$

$f \in \mathcal{D}'(\mathbb{R})$

$\forall x \in \mathbb{R}, \quad f'(x) - 3f(x) = (2ax + 2b + a)e^{2x} - 3(ax+b)e^{2x}$

$= (-ax - b + a)e^{2x}$

on prend $\begin{cases} -a = 1 \\ -b + a = 0 \end{cases}$

$\Leftrightarrow \begin{cases} a = -1 \\ b = -1 \end{cases}$

$x \longmapsto -(x+1)e^{2x}$ est solution particulière.

③ $\mathcal{S} = \{ x \longmapsto \lambda e^{3x} - (x+1)e^{2x}, \lambda \in \mathbb{R} \}$

Exemple $y' - 3y = e^{3x}$

① $\mathcal{S}_0 = \{ x \longmapsto \lambda e^{3x}, \lambda \in \mathbb{R} \}$

② solution particulière

on prend $a \in \mathbb{R}$

$f : x \longmapsto axe^{3x}$

$f \in \mathcal{D}'(\mathbb{R})$

$f : x \longmapsto ae^{3x} \in \mathcal{S}_0$

$\forall x \in \mathbb{R}, \quad f'(x) = 3ae^{3x}$

$f'(x) - 3f(x) = 3ae^{3x} - 3ae^{3x}$

$= 0$

$\forall x \in \mathbb{R}, \quad f'(x) - 3f(x) = (3ax + a)e^{3x} - 3axe^{3x}$

$= ae^{3x}$

on prend $a = 1$ alors f solution particulière

③ $\mathcal{S} = \{x \mapsto \lambda e^{3x} + xe^{3x}, \lambda \in \mathbb{R}\}$

$= \{x \mapsto (\lambda + x)e^{3x} \quad \lambda \in \mathbb{R}\}$

Cas général $\begin{cases} y' - 3y = P(x)e^{3x} \end{cases}$

on pose $f : x \mapsto Q(x)e^{3x} \in \mathcal{D}^1(\mathbb{R})$

$\forall x \in \mathbb{R} \quad f'(x) - 3f(x) = (3Q(x) + Q'(x))e^{3x} - 3Q(x)e^{3x}$

$= Q'(x)e^{3x}$

on prend Q tel que $\forall x \in \mathbb{R}, \ Q'(x) = P(x)$

Variation de la constante

on pose $f : x \mapsto \lambda(x)e^{3x}$ avec $\lambda \in \mathcal{D}^1(\mathbb{R})$

$\forall x \in \mathbb{R} \quad f'(x) - 3f(x) = \lambda'(x)e^{3x}$

on cherche λ tel que $\forall x \in \mathbb{R} \quad \lambda'(x)e^{3x} = e^{3x}$

donc $\lambda'(x) = 1$

on prend $\lambda : x \mapsto x$

Variation de la constante pour le second membre xe^{2x}.

$\forall x \in \mathbb{R} \quad f(x) = \lambda(x)e^{3x} \quad f \in \mathcal{D}^1(\mathbb{R})$

$\forall x \in \mathbb{R}, \quad f'(x) - 3f(x) = \lambda'(x)e^{3x}$

on cherche λ tel que $\forall x \in \mathbb{R} \quad \lambda'(x)e^{3x} = xe^{2x}$

$\lambda'(x) = xe^{-x}$

IPP

primitive $x \mapsto (ax + b)e^{-x}$

etc... (plus long)

Exemple $y' - 3y = 2e^{3x} + (4 - x)e^{2x}$

Superposition